Buckling of Circular Cylindrical Shells Having Axisymmetric Imperfection Distributions

R. C. TENNYSON,* D. B. MUGGERIDGE,† AND R. D. CASWELL‡
University of Toronto, Institute for Aerospace Studies, Toronto, Ontario, Canada

A study has been made of the effect of axisymmetric imperfection distributions on the buckling behavior of circular cylindrical shells under axial compression. The imperfection profiles considered include a uniform distribution of sine waves, groups of constant amplitude sine waves of varying wavelength and, of particular practical interest, random distributions. In each case, test models were constructed from a photoelastic liquid epoxy using the spin-casting technique, and the experimental buckling data was compared with available theory. For the uniform and mixed mode distributions, it was observed that a critical axisymmetric wavelength existed that yielded a minimum buckling load for a given value of imperfection amplitude, consistent with the predictions of Koiter's extended theory. Using Fourier series analysis to estimate a power spectral density, it was concluded that for the range of random axisymmetric imperfections considered, a critical frequency component governed buckling as predicted by theory. Furthermore, it was found that a lower bound estimate of the buckling load was obtained based on the root-mean-square imperfection amplitude using an exponential-cosine autocorrelation peaked at the critical frequency.

Nomenclature

A_j,B_j = Fourier coefficients nondimensionalized by $\bar{t}$
c = $[3(1 - \nu^2)]^{1/2}$
E = modulus of elasticity
h = $t/\bar{t}$
K = p/q_0
ℓ = shell length
L = $\ell q_0/R$
l_x = $\ell/m = \pi R/2p$, axial half-wave length
$l_{x_{cl}}$ = $\pi R/q_0$
l_{xj} = imperfection half-wave length corresponding to jth Fourier term
m,n = number of half-waves in the axial and circumferential directions, respectively
p = $m\pi R/2\ell$
q_0 = $(R/t)^{1/2}[12(1 - \nu^2)]^{1/4}$, the classical axisymmetric buckling mode wave number
R = shell radius measured to the median surface
$S(\omega)$ = power spectral density
t = shell wall thickness
$\bar{t}$ = average shell wall thickness
$w,\tilde{w}$ = radial deviation of the outer and median surfaces, respectively
$W,\tilde{W}$ = $w/\bar{t}$, $\tilde{w}/\bar{t}$, respectively
x,y = axial and circumferential coordinates, respectively
X = xq_0/R
δ = peak axisymmetric imperfection amplitude measured on external surface
ζ = rms deviation of median surface/average shell wall thickness
λ = theoretical buckling stress of axisymmetric imperfect circular cylinder/classical buckling stress
λ^* = σ_{cr}/σ_{cl}
μ = maximum deviation of median surface/average shell wall thickness
ν = Poisson's ratio
σ_{cl} = Et/Rc, classical compressive buckling stress for a perfect circular cylindrical shell
σ_{cr} = experimental cylinder buckling stress
τ = $n(t/Rc)^{1/2}$
ω_j = $l_{x_{cl}}/l_{xj}$, nondimensional spatial frequency

Subscripts

cl = classical
cr = critical
rms = root mean square

Received July 20, 1970; revision received November 12, 1970. The authors wish to gratefully acknowledge the financial assistance of the National Research Council of Canada (NRC Grant 2783) and NASA [Grant 52-026-(011) Sup #2] which made this work possible.

* Associate Professor. Member AIAA.
† Research Engineer, Fleet Manufacturing Ltd., Fort Erie, Ontario.
‡ Research Engineer, SPAR Aerospace, Malton, Ontario.

I. Introduction

IN the design of circular cylindrical shells, buckling load calculations play a primary role in determining the load carrying capacity of the structure under compressive loading. The most recent design manuals[1,2] suggest that classical buckling loads should be reduced by an empirical "correlation" factor which is based on an accumulation of circular cylinder buckling data through which a conservative design curve was fitted. Since these load reductions are quite large for $R/t \geq 100$, it is of importance to determine if reliable buckling load predictions can be established using other criteria. If it can be demonstrated that analytical methods are available for accurately predicting buckling loads consistent with experimental observations, it then becomes possible to consider various means of increasing the critical loads by improved design, manufacturing, and handling techniques.

At present, the effects of cylinder edge constraints on reducing the classical stress are well known,[3,4] although the clamped case which is most often encountered in practice accounts only for a 10% reduction. Generally, it is recognized that the dominant factor in reducing the buckling load of circular cylinders significantly below the classical value results from the presence of shape imperfections in the shell wall. Clearly, if these imperfection distributions are random in nature, it is exceedingly difficult to formulate a rational design criteria in the absence of a random model theory and a practical method of determining the distribution of imperfections on a shell structure of reasonable size. Consequently, it is imperative to determine both theoretically and experimentally what imperfection distributions and modal components effectively control buckling of cylinders under axial compression. Based on Koiter's theory[3] for a uniform distribution of simple sine waves, extended solutions have been obtained and compared with experimental results.[4] It was observed that the buckling load reductions were dependent on imperfection amplitude and wavelength. In particular, a critical wavelength was found to yield a minimum buckling load for a given value of imperfection amplitude. Other investigators have also demonstrated experimentally[5] that certain modal components for general shape imperfections appear to affect the buckling load. At present, theoretical analyses[6,7] are available which describe the effect of random axisymmetric shape imperfections on the buckling behavior of circular cylinders. Although it may be argued that existing theory is of limited practical interest, it is first necessary to isolate the major parameters that can be shown to control buckling both theoretically and experimentally. With this objective in mind, the present work contains a study of the effect of various axisymmetric imperfection distributions on the buckling behavior of circular cylinders.

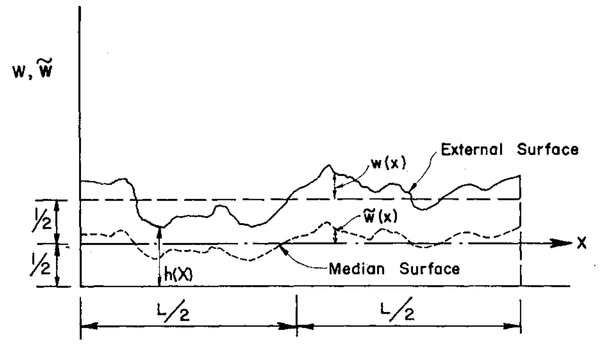

Fig. 1 Random axisymmetric imperfection distribution.

II. Basic Equations

Uniform Distribution

The problem of determining the effect of a uniform axisymmetric distribution of imperfections having the form of a simple sine wave was first analyzed by Koiter[8] for the case of the imperfection wavelength equal to the classical axisymmetric buckling mode. Although an asymptotic solution was obtained for small values of the imperfection amplitude μ, Koiter subsequently formulated a special theory[3] using a Galerkin procedure to obtain an upper bound buckling load solution for finite values of the imperfection amplitude and arbitrary wavelengths. The following eigenvalue equation was obtained:

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{1}$$

where

$$A_1 = -512\ K^6Q^2 \tag{2a}$$

$$A_2 = 64K^4Q^4 + 1024K^8 + 128K^4BQ^2 + 128c\mu\tau^2K^4Q^2 \tag{2b}$$

$$A_3 = -16K^2BQ^4 - 256K^6B - 8K^2Q^2B^2 - 16c\mu\tau^2K^2Q^2B + 512c\mu\tau^2K^6B \tag{2c}$$

$$A_4 = Q^4B^2 + 16K^4B^2 - 64c\mu\tau^2K^4B^2 + 64(c\mu)^2K^4Q^2\tau^4B^2H \tag{2d}$$

$$Q = 2K^2 + \tau^2, \quad B = 16K^4 + 1 \tag{2e}$$

$$H = 1/Q^2 + 1/(18K^2 + \tau^2)^2 \tag{2f}$$

for an imperfection of the form $\tilde{W} = -\mu\cos(\pi x/l_x)$. Koiter solved Eq. (1) for the particular case when $K = \frac{1}{2}$, which corresponds to the axisymmetric imperfection wavelength equal to the classical axisymmetric buckling mode of the perfect circular cylindrical shell. The general solution[4] of Eq. (1) was found to yield a critical imperfection wavelength corresponding to a minimum value of the buckling load for a given value of imperfection amplitude. Only as $\mu \to 0$ (i.e.; the asymptotic solution) does $K = \frac{1}{2}$ give the minimum buckling load.[4] For this case,[3] Eq. (1) reduces to

$$2(1 - \lambda)^2 - 3c\mu\lambda = 0 \tag{3}$$

in the vicinity of $\lambda \to 1$, which is asymptotically equivalent to

$$\lambda = 1 - (3c/2)^{1/2}\mu^{1/2} \tag{4}$$

Random Axisymmetric Distribution

Amazigo[6] determined an asymptotic formula for the buckling load of a long circular cylindrical shell containing an axisymmetric distribution of homogeneous random geometric imperfections in shape. The following equation was derived neglecting the effect of end constraints and assuming small mean square values of the imperfection:

$$\lambda = 1 - \{[9\pi c^2/2(2)^{1/2}]S(1)\}^{2/7} \tag{5}$$

where $S(1)$ corresponds to the power spectral density of the imperfection spectrum $S(\omega)$ evaluated at the frequency of the classical axisymmetric buckling mode, i.e.;

$$\omega = l_{x\mathrm{cl}}/l_x = 1 \tag{6}$$

A modified form of Eq. (5) which is asymptotically equivalent can be obtained by including higher order terms in Amazigo's analysis. The proposed equation[9] extends the asymptotic range of Eq. (5) to finite values of the mean square value of the imperfection,

$$(1 - \lambda)^{7/2} = [9\pi c^2/2(2)^{1/2}]S(1)\lambda^2 \tag{7}$$

Estimate of the Power Spectral Density

In general, the power spectral density $S(\omega)$ of the axisymmetric imperfection distribution $W(X)$ for a long cylinder can be obtained by computing the autocorrelation from a spatial record of $W(X)$ and then taking the Fourier transform of the autocorrelation.[10] However, for cylinders having lengths of approximately ten times the bandwidth of the imperfection wavelengths being studied, it was found that the spatial record $W(X)$ obtained for each generator was insufficient for determining $S(\omega)$ using the above procedure due to frequency limitations. A value of $S(\omega)$ characterizing the complete cylinder could be obtained by analyzing a record of $W(X)$ comprised of many generator profiles recorded in series, although erroneous frequency data would result. In any case, $S(\omega)$ was required for each generator to assess the degree of asymmetry in the random axisymmetric imperfection distribution and thus Fourier series analysis was employed to estimate the power spectral density in the form of a line spectrum.

If the deviation of the shell surface due to a random axisymmetric imperfection distribution is given by $W(X)$ (Fig. 1) where

$$\frac{1}{L}\int_0^L W(X)dX = 0 \tag{8}$$

then the Fourier series representation of $W(X)$ is

$$W(X) = \sum_{j=0}^{\infty} A_j \cos\omega_jX + B_j \sin\omega_jX \tag{9}$$

where

$$X = xq_0/R, \quad \omega_j = \pi R/l_{xj}q_0, \quad L = \ell q_0/R$$

$$A_j = \frac{2}{L}\int_{-L/2}^{L/2} W(X)\cos\omega_jXdX \tag{10}$$

$$B_j = \frac{2}{L}\int_{-L/2}^{L/2} W(X)\sin\omega_jXdX$$

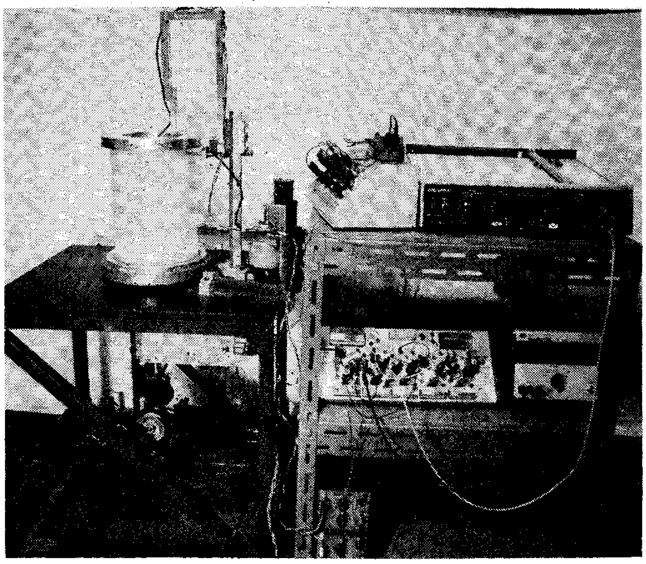

Fig. 2 Cylinder profile measuring apparatus.

By virtue of Eq. (8), $A_0 = 0$. In complex form, Eq. (9) can be rewritten as

$$W(X) = \sum_{j=-\infty}^{\infty} C_j e^{i\omega_j X} \tag{11}$$

where

$$C_j = \frac{1}{L}\int_{-L/2}^{L/2} W(X)e^{-i\omega_j X}dX$$
$$= \tfrac{1}{2}(A_j - iB_j) \tag{12}$$

The mean square value of the imperfection is given by,

$$\overline{W^2(X)} = \frac{1}{L}\int_{-L/2}^{L/2} W^2(X)dX$$
$$= \sum_{j=0}^{\infty} \frac{1}{2}(A_j^2 + B_j^2) \tag{13}$$
$$= 2\sum_{j=0}^{\infty} |C_j|^2 = \sum_{j=-\infty}^{\infty} |C_j|^2$$

since $4|C_j|^2 = A_j^2 + B_j^2$.

The power spectral density is related to the mean square value by the following relationship,

$$\overline{W^2(X)} = 2\int_0^{\infty} S(\omega)d\omega$$
$$= \int_{-\infty}^{\infty} S(\omega)d\omega \tag{14}$$

where $S(\omega)$ defines a two-sided spectrum. By comparing Eqs. (13) and (14), the power spectrum can be regarded in the limit as being equal to

$$S(\omega) = \lim_{\omega_0 \to 0} \frac{|C_j|^2}{\omega_0}$$
$$= \lim_{\omega_0 \to 0} \left(\frac{A_j^2 + B_j^2}{4\omega_0}\right) \text{ where } \omega_0 = 2\pi/L \tag{15}$$

It should be noted that the application of Eq. (15) to estimate spectral density based on sample records drawn from a random process can lead to questionable results. However, for the shells considered in this investigation, complete profiles were analyzed to obtain $S(\omega)$. Thus, the power spectral density of the median surface (for a constant thickness shell) evaluated at the critical frequency corresponding to the classical axisymmetric buckling mode (i.e., $l_{xj} = l_{xcl}$; $\omega = 1$) can be estimated from the following relation:

$$S(1) \simeq (A_{cr}^2 + B_{cr}^2)\ell q_0/8\pi R \tag{16}$$

For the special case of the random axisymmetric imperfection present only on one shell surface, which results in an axisymmetric thickness variation, i.e.;

$$h(X) = 1 + W(X) \tag{17}$$

the Fourier coefficients describing the median surface must be altered to one-half the value of the coefficients characterizing the external surface profile. Hence, Eq. (16) must be modified to,

$$S(1) \simeq (A_{cr}^2 + B_{cr}^2)\ell q_0/32\pi R \tag{18}$$

Substituting either Eq. (16) or Eq. (18) into Eq. (7) yields the modified buckling load solution as derived from Amazigo's analysis. An alternative formulation can be obtained in terms of the root-mean-square imperfection amplitude by assuming a power spectral density function. Of particular interest is the spectrum corresponding to an exponential-cosine autocorrelation which has been studied both by Amazigo[6] and Fersht.[7] The latter author employed a Lyapunov method to obtain a sufficient condition for the stability of a circular cylinder containing a random axisymmetric distribution of shape imperfections. Since the Lyapunov technique always yields a conservative estimate, it is of interest to compare Fersht's solution with the results of Amazigo. In both cases, the parameters defining the exponential-cosine autocorrelation were selected to peak the distribution at the critical frequency ($\omega = 1$). Hence, the appropriate form for $S(1)$ is given by[6]

$$S(1) = \zeta^2\beta(1 + \beta^2 + \gamma^2)/\pi[1 + 2(\beta^2 - \gamma^2) + (\beta^2 + \gamma^2)^2] \tag{19}$$

where $\beta = 0.2$ and $\gamma = 1$. Substituting Eq. (19) into Eq. (7) yields,

$$(1 - \lambda)^{7/2} = [9c^2/2(2)^{1/2}]2.52\,\zeta^2\lambda^2 \tag{20}$$

Thus, Eq. (20) describes a conservative estimate of the buckling load as a function of the root-mean-square imperfection due to the localized peak in the spectrum at $\omega = 1$.

III. Experiment—Fabrication and Test Procedure

All circular cylindrical shell test models containing the various axisymmetric imperfection distributions were manufactured from a photoelastic liquid plastic using the spin-casting technique.[11] Except for one case, each cylinder contained an imperfection profile cut only on one surface. In the first test series, geometrically near-perfect cylinders were initially cast and the inner wall machined to a prescribed profile. Subsequently, it was decided to fabricate some imperfection distributions on the outer surface for several cylinders. This was easily accomplished by cutting the desired imperfection wave shape on the inside wall of the casting form and then spin-casting the liquid plastic. In one particular case in which a constant wall thickness was required, an additional machining operation was performed in which the profile of the casting form was cut on the inner surface of the spun-cast shell. Although some difficulty was encountered in extricating the cylinder models with outer surface shape imperfections from the casting form, separation was achieved by using a hydraulic press with a plunger.

In order to fabricate the axisymmetric imperfections into the circular cylinders, a metal template containing the desired wave forms was used in conjunction with a hydraulic tracer-tool apparatus. The profiles were constructed on the templates by means of a circular cam with its center of rotation offset to provide the appropriate amplitudes. The eccentric cam was mounted on a motor shaft such that continuous contact was maintained with a spring-loaded cutting tool. Various feed and speed combinations of a lathe were used together with a particular speed of the motor driving the cutting tool to establish an imperfection wavelength. Once all speed combinations were determined, the desired sinusoidal imperfection profile of arbitrary amplitude and wavelength was cut into the metal template.

A more complicated procedure was employed, however, to produce the 'random' axisymmetric imperfection distributions. Initially, this shell series called for a shape imperfection specified by the sum of three cosine functions. In order to achieve this, the desired profile was plotted and a close fit to it was obtained by using several cosine terms of different amplitudes and frequencies, each of which resulted in an individual template being made. The final profile was then manufactured by superimposing the various components and carefully machining them into one template. Random deviations from the prescribed distribution accrued during the final stage of constructing the master template, with some high frequency components being replaced by local flat areas. The main reason for fabricating a profile comprised primarily of three cosine terms was to ensure that frequency components close to the critical value (corresponding to the classical axisymmetric buckling mode wavelength) were present in the distribution. However, a true random variation was obtained for the last shell (AD 12) by superimposing a random cut on the inner wall of the form without the use of a template.

After removing the cylinders from the casting form, thickness measurements were made at discrete points around the circumference at both ends. Each shell was subsequently positioned in a rotation apparatus, as shown in Fig. 2, and the inner, outer, and median surface profiles were determined using two low-pressure, linear contacting displacement transducers with their outputs recorded on an X-Y plotter. Figure 3 illustrates median surface profiles obtained from typical cylinders having a uniform, grouped, and "random" distribution of axisymmetric imperfections. The measured thickness at the shell ends served as a reference for determining absolute variations along each generator. Prior to each traverse, an integrator in combination with a digital voltmeter was initially set to zero and thus an average shell wall thickness was obtained. In each case, a cylinder generator was selected at random from the X-Y plotter traces and digitized into approximately 300 data points which were subsequently analyzed by means of a Fourier series computer program to yield the complex coefficients given by Eq. (12). Hence an estimate of the power spectral density for each cylinder was computed using either Eq. (15) or Eq. (16). Typical results obtained for C_j and $S(\omega)$ are shown in Fig. 4. For comparison purposes, $S(\omega)$ for cylinders AD 2 (constant amplitude variable frequency) and AD 12 is shown in Fig. 5 to indicate the significant differences between test specimens. In order to obtain average values of the Fourier coefficients and power spectral densities describing the imperfection distribution, simultaneous recording of the probes' output signal on magnetic tape was required for each generator. Thus, the asymmetry in the imperfection distribution was determined for two cylinders by digitizing all generator data and plotting $S(\omega)$ normalized by the average value $\overline{S(\omega)}$ as a function of the circumferential coordinate (refer to Fig. 6).

In total, twenty-three circular cylindrical shells were tested in axial compression. Two of the cylinders were geometrically 'near-perfect' and served as reference shells to determine the modulus of elasticity of the epoxy plastic and, at the same time, to provide a measure of the load reduction caused by the clamped edge constraint common to each shell. Thus twenty-one axisymmetric imperfect test models were studied, nine of which contained a uniform distribution of sine waves (the results are presented in Ref. 4), five consisted of various distributions of constant amplitude sine waves of variable

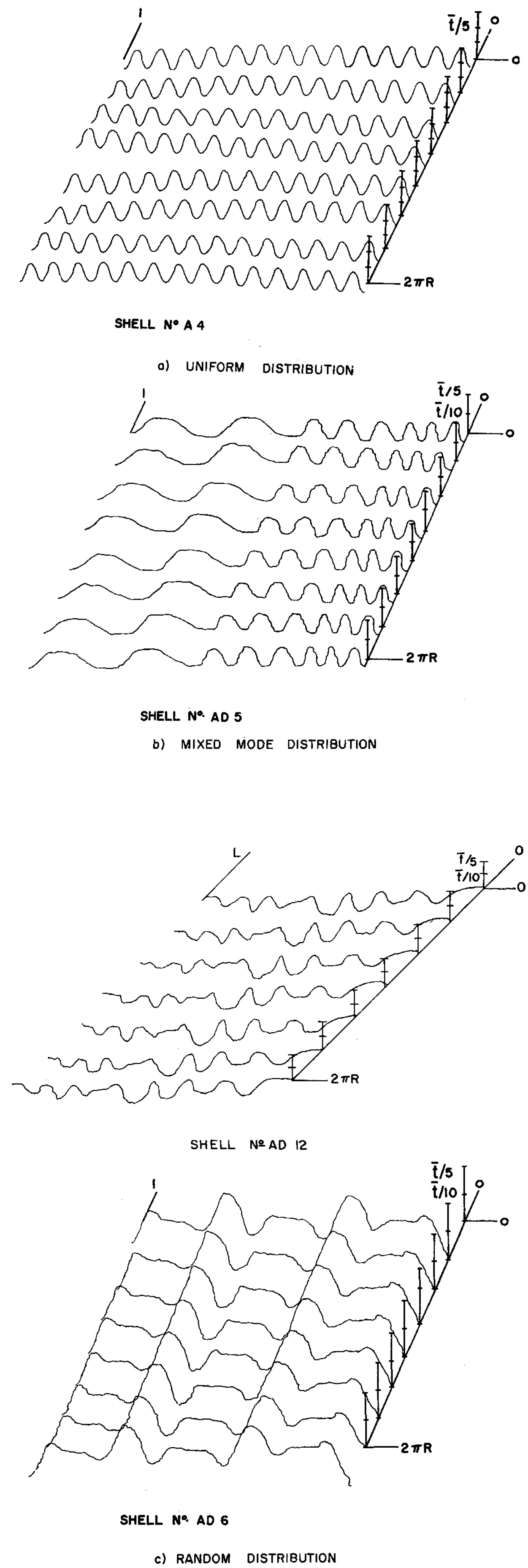

Fig. 3 Median surface profiles of axisymmetric imperfect circular cylindrical shells.

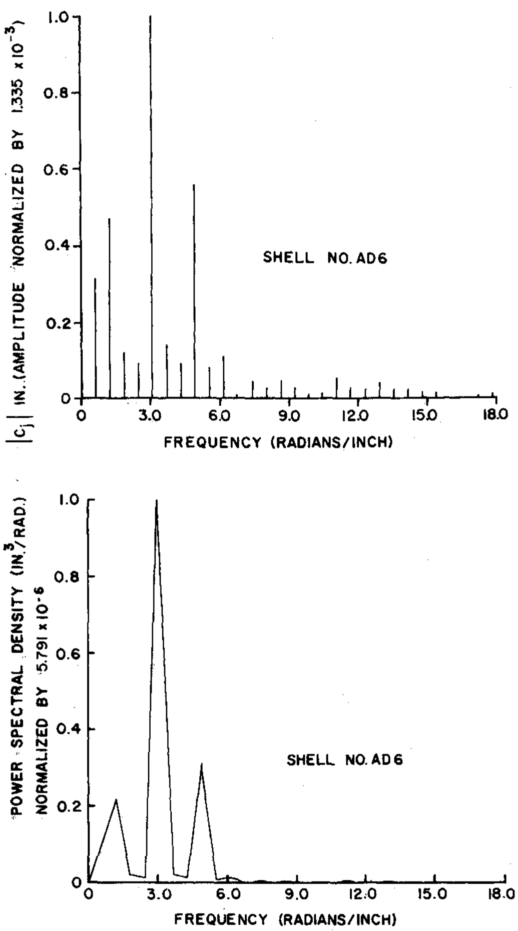

Fig. 4 Estimated frequency and power spectra of axisymmetric imperfections.

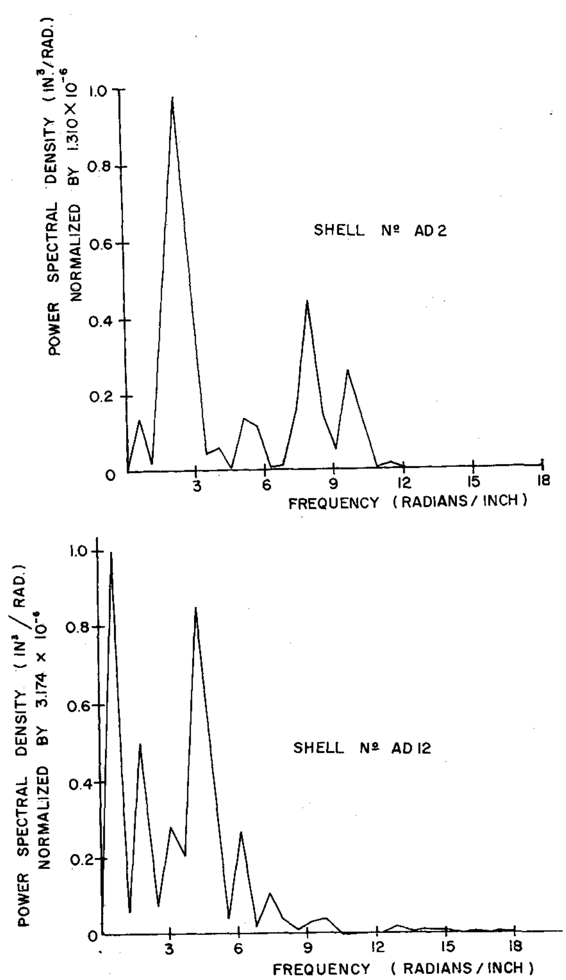

Fig. 5 Estimated power spectrum of axisymmetric imperfections.

frequency (Table 1) and the remaining seven cylinders had a 'random' distribution (Table 2). Each cylinder was tested in an electrically driven, four-screw, constant end displacement compression machine. The photoelastic property of the shells was used to ensure a uniform distribution of the applied stress. In addition, since all cylinders behaved elastically, each was tested several times and found to yield repeatable buckling loads. The results of this investigation are discussed in the next section.

IV. Discussion of Experimental Results

Mixed Group Discussion

Based on the results obtained for the uniform distribution of axisymmetric imperfections,[4] it was decided to determine the extent to which the critical wavelength as obtained from the solution of Eq. (1) for a given shell geometry would dominate the buckling behavior of a cylinder containing a mixed group of constant amplitude sine waves of varying frequency.

Five cylinders, each containing a constant amplitude sine wave imperfection distribution comprised of three distinct wavelengths cut on the inside surface, were tested in axial compression. Three of the test models varied only in the location of the critical axisymmetric imperfection wavelength along the generator, while the remaining two cylinders differed in the number of critical waves centered at mid length (refer to Table 1). It can be seen from Fig. 7 that wavelength location had only a small effect on the buckling load. By increasing the number of critical imperfection wavelengths to the point of a uniform distribution, it was observed that except for small differences, the buckling load reduction was essentially governed by the presence of the critical frequency corresponding to the solution of Eq. (1).

Random Distribution

A study of the effect of random axisymmetric imperfection distributions on the buckling behavior of seven circular cylinders (refer to Table 2) was made to determine if a critical frequency component governed the load reduction. It was found that the best consistent estimate of $S(1)$ was obtained by choosing the largest value of C_j within a narrow bandwidth of $\omega_j = 1$. Although the various distributions were not strictly 'random' in a statistical sense, it was felt that the profiles were sufficiently general to include a reasonable frequency bandwidth of imperfections. Two profiles of the mixed group distributions were also analyzed in a similar manner.§

A comparison of the buckling load results with Amazigo's asymptotic theory[6] given by Eq. (5) is contained in Fig. 8. In general, asymptotic solutions are valid for very small values of imperfection amplitudes and are limited to a range of λ close to unity. As noted earlier, modified forms of the asymptotic equations can be obtained which provide reasonable estimates of the critical load parameter up to values of μ of the order of the shell wall thickness.[3,4,12] Hence, a modified form of Eq. (5) given by Eq. (7) is also plotted in Fig. 8 for

§ For both shells AD2 and AD3, a choice between two comparable values of C_j was required within the frequency bandwidth. The selection was made using Koiter's theory[4] assuming $\mu_j = C_j$, $K_j = \ell_{xcl}/\ell_{xj}$ and determining which component gave the lowest value of λ.

Table 1 Properties of shells containing groups of axisymmetric imperfections

Shell no.	R, in.	$\bar{t}$, in.	$R/\bar{t}$	ℓ, in.	δ, in.	$\delta/2\bar{t}$	K	λ^*
AD1	3.92	0.0200	198	11.0	0.0021	0.053	0.717	0.669
							0.466	
							0.178	
AD2	3.92	0.0201	196	11.0	0.0019	0.047	0.721	0.706
AD3	3.92	0.0204	192	11.0	0.0023	0.056	0.468	0.689
							0.179	
AD4	3.92	0.0205	191	11.0	0.0018	0.044	0.729	0.670
							0.473	
							0.181	
AD5	3.92	0.0209	187	11.0	0.0021	0.050	0.735	0.628
							0.477	
							0.183	

$E = 3.94 \times 10^5$ psi
$\nu = 0.40$
$\ell_x = 0.343$ in., 0.528 in., 1.378 in.

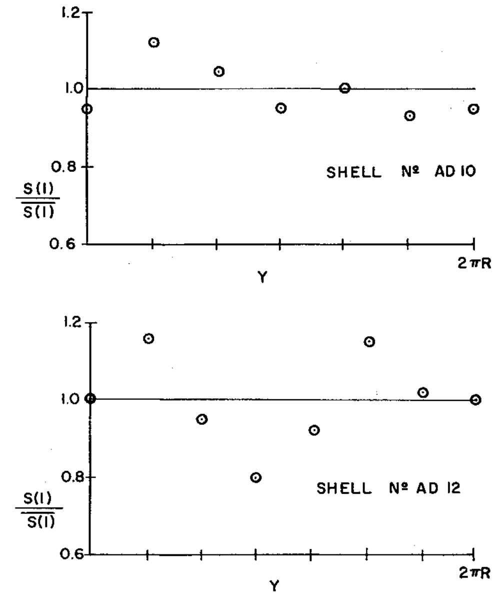

Fig. 6 Circumferential PSD variation.

comparison purposes. In general, theory and experiment agree quite well. However, it should be noted that of the nine cylinders tested, only one (AD 11) was a constant thickness model as assumed in the theory.[6] For the remaining cylinders, imperfection profiles were cut either on the inner (shells AD 2,3,6,7,8) or outer (shells AD 9,10,12) surfaces, resulting in an axisymmetric thickness variation given by Eq. (17). The considerable difference in buckling loads due to imperfection distributions being either on the inner or outer surfaces was primarily a result of thickness variations along the generator. This can readily be seen by comparing the buckling results of shells AD 9,10 and 12 which had external profiles. It was observed that the minimum thickness for shell AD12 was much less than the average wall thickness as compared to the other two cylinders. In general, it was also found that the difference between the minimum and average shell wall thicknesses for a cylinder with an imperfection profile on the inner surface was three times larger than that for a similar shell (i.e., having the same value of $\bar{t}$) with the same imperfection profile cut on the outer surface. It should also be noted that although some circumferential variation in the power spectral density estimate $S(1)$ occurred (Fig. 6), little effect on the buckling load comparison resulted due to the small exponent appearing in Eqs. (5) and (7).

A comparison of the cylinder buckling data with the theory of Amazigo[6] and Fersht[7] using the rms deviation of the imperfection distribution is given in Fig. 9. Employing a Lyapunov method to obtain a sufficient condition for the stability of a circular cylinder containing a random axisymmetric shape imperfection, Fersht obtained an estimate of the buckling loads by assuming an exponential-cosine autocorrelation peaked at the frequency $\omega_j = 1$, and a Gaussian distribution of initial imperfections. The results shown in Fig. 9 indicate that a very conservative stability boundary was obtained, although the test cylinder parameters did not correspond to those values used in the calculation. On the other hand, the comparison between experiment and Amazigo's theory [given by Eq. (20)] is rather good, despite the scatter in the data which is primarily due to the axisymmetric thickness variations which are not accounted for in the analysis. It would thus appear that the choice of the exponential-cosine autocorrelation peaked at the critical frequency ($\omega_j = 1$) provides an excellent estimate of the buckling load in terms of the rms amplitude of the imperfection distribution. The fact that the actual PSD distributions in the test models differed from that described by Eq. (19) does not seriously affect the comparison, providing the imperfection components (C_j) are of the same order as (or less than) the value at $\omega_j = 1$. This can be justified on the basis that the solutions for both the uniform and random axisymmetric imperfection distributions, given by Eq. (1) for $K = \frac{1}{2}$ and Eq. (20), respectively, yield comparable buckling load reductions. Consequently, conservative buckling load estimates based on Eq. (19) should result.

It is evident from the results achieved to date that considerable effort can be saved in estimating cylinder buckling loads using the rms deviation of the imperfection distribution without analyzing the power spectrum. However, the PSD technique gives important information on the frequency components, which can be of particular significance in refining manufacturing and handling procedures in order to increase design buckling load estimates.

Table 2 Properties of shells containing "random" axisymmetric imperfections

Shell no.	R, in.	$\bar{t}$, in.	$R/\bar{t}$	ℓ, in.	δ_{rms}, in.	ζ	$\lvert C\rvert_{cr} \times 10^{-3}$, in.	$S(1) \times 10^{-3}$	λ^*
[a]AD6	3.91	0.0299	131	10.2	0.0025	0.042	0.748	1.324	0.620
[a]AD7	3.84	0.0386	100	10.2	0.0025	0.032	0.746	0.704	0.710
[a]AD8	3.85	0.0213	181	10.2	0.0025	0.059	0.708	2.788	0.564
[b]AD9	3.85	0.0286	135	10.2	0.0023	0.040	0.617	1.008	0.760
[b]AD10	3.86	0.0244	158	10.2	0.0026	0.053	0.734	2.191	0.715
[c]AD11	3.86	0.0230	168	10.2	0.0023	0.100	0.675	8.340	0.530
[b]AD12	3.86	0.0269	144	10.2	0.0029	0.053	0.912	2.575	0.556
[a]AD2	3.92	0.0201	196	11.0	0.0017	0.042	0.407	1.139	0.706
[a]AD3	3.92	0.0204	192	11.0	0.0017	0.041	0.353	0.830	0.689

$E = 3.94 \times 10^5$ psi
$\nu = 0.40$

[a] Imperfection profile on inside wall only.
[b] Imperfection profile on outside wall only.
[c] Constant wall thickness cylinder.

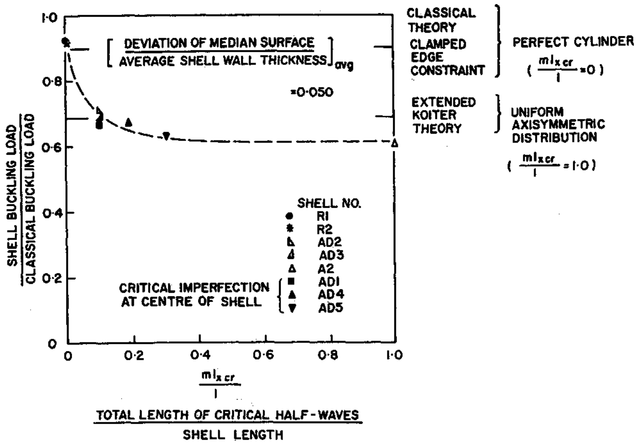

Fig. 7 **Buckling load ratio of a circular cylindrical shell containing groups of axisymmetric imperfections vs critical half-wave parameter for a particular value of axisymmetric imperfection amplitude.**

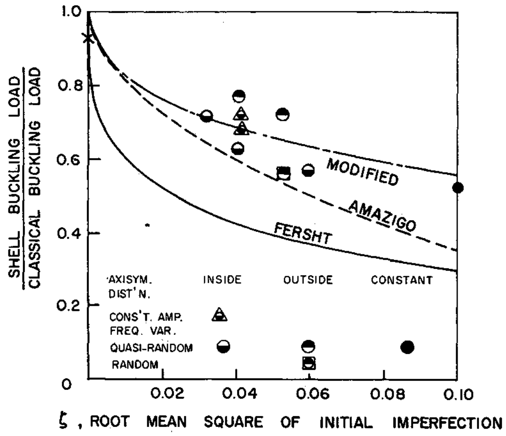

Fig. 9 **Stability boundary for a cylindrical shell with random axisymmetric imperfections.**

V. Conclusions

It has been shown that for circular cylindrical shells having a uniform axisymmetric distribution of imperfections, a critical wavelength exists leading to a minimum buckling load for a particular imperfection amplitude and that buckling loads were reduced by increasing values of the imperfection amplitude for a given wave number, as predicted by Koiter. In addition, it was demonstrated that the presence of the critical imperfection wavelength in shells containing groups of constant amplitude axisymmetric imperfections of different frequencies resulted in buckling loads corresponding to that obtained for a shell containing a uniform distribution of axisymmetric imperfections of the critical frequency with the same amplitude. For the case of random axisymmetric imperfection distributions, results have been obtained which indicate that the buckling behavior is essentially governed by the power spectral density of the imperfections evaluated within a narrow bandwidth of the critical frequency given by $\omega_j = 1$. It was also found that a good correlation existed between buckling loads and the root-mean-square deviation of the imperfection distribution assuming an exponential-cosine autocorrelation peaked at $\omega_j = 1$. Consequently, it would appear that this last series of shells yielded data which tends to validate the general random imperfection approach for the determination of buckling loads of practical shell structures

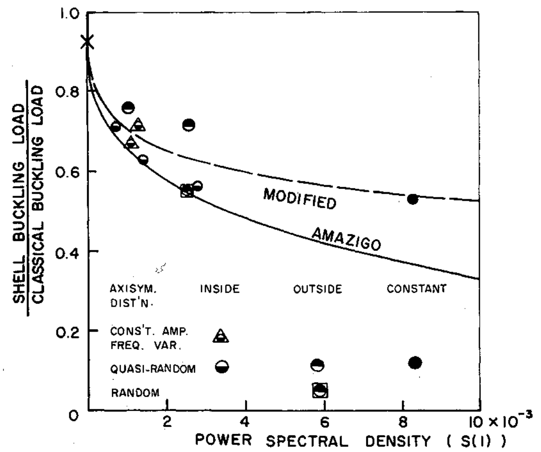

Fig. 8 **Comparison of axisymmetric imperfect shell buckling loads with Amazigo's theory.**

References

[1] Baker, E. H. et al., "Shell Analysis Manual," CR-912, April 1968, NASA.

[2] "Buckling of Thin-Walled Circular Cylinders," *Space Vehicle Design Criteria (Structures)*, NASA SP-8007, Aug. 1968.

[3] Koiter, W. T., "The Effect of Axisymmetric Imperfections on the Buckling of Cylindrical Shells Under Axial Compression," *Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings*, B66, 1963, pp. 265–279.

[4] Tennyson, R. C. and Muggeridge, D. B., "Buckling of Axisymmetric Imperfect Circular Cylindrical Shells under Axial Compression," *AIAA Journal*, Vol. 7, No. 11, Nov. 1969, pp. 2127–2131.

[5] Arbocz, J. and Babcock, C. D., "Experimental Investigation of the Effect of General Imperfections on the Buckling of Cylindrical Shells," CR-1163, Sept. 1968, NASA.

[6] Amazigo, J. C., "Buckling Under Axial Compression of Long Cylindrical Shells with Random Axisymmetric Imperfections," *Quarterly of Applied Mathematics*, Vol. XXVI, No. 4, Jan. 1969, pp. 537–566.

[7] Fersht, R. S., "Almost Sure Stability of Long Cylindrical Shells With Random Imperfections," CR-1161, 1968, NASA.

[8] Koiter, W. T., "On the Stability of Elastic Equilibrium," Thesis, 1945, Delft, Amsterdam; also available as TTF-10, 833, March 1967, NASA.

[9] Budiansky, B. and Amazigo, J. C., private communication, 1970, Harvard Univ.

[10] Bendat, J. S. and Piersol, A. G., "Measurement and Analysis of Random Data," Wiley, New York, 1966.

[11] Tennyson, R. C., "Photoelastic Circular Cylinders in Axial Compression," STP 419, 1967, American Society for Testing and Materials, pp. 31–46.

[12] Hutchinson, J. W., Tennyson, R. C., and Muggeridge, D. B., "The Effect of a Local Axisymmetric Imperfection on the Buckling Behaviour of a Circular Cylindrical Shell Under Axial Compression," AIAA Paper 70-103, New York, 1970.